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A Study of Different Methods of Interpolation As Applied To The Sine Function

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A STUDY OF DIFFERENT METHODS OF INTERPOLATION

AS APPLIED TO THE SINE FUNCTION

being

A thesis presented to the Graduate Faculty

of the Fort Hays Kansas State College in

partial fulfillment of the requirements for

the Degree of Master of Science

by

Erwin L. Penner, A. B.

Friends University

Date July 22, 1953   Approved Emmet Stopher

Major Professor

Ralph V. Coder

Chairman Graduate Council

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INTRODUCTION

Mathematics is a science complete in itself. In studying mathematics the use of some kind of mathematical tables will be needed for certain types of courses. The tables available will not always be sufficient to give the desired results. This will necessitate some kind of interpolation to give the information wanted. Interpolation is the art or science of "reading between the lines of mathematical tables." According to the definition, interpolation is the insertion of mathematical terms according to the law of the function involved.

The kind of interpolation usually used, since it is the common one, is linear or straight line interpolation. This method uses the value directly above and below the desired result, finding the difference between these tabulated values, taking the proportional part thereof, and adding this to the smaller given value. This result is usually sufficiently accurate for work on the high school level since accuracy to four decimal points is wanted. The approximate answer thus obtained is due partly to the fact that when these two values are so used they do not necessarily trace the function involved but divert it into a straight line.

When the study of this problem was undertaken it was the normal assumption that parabolic and cubic interpolation would give a more accurate result than the common conventional straight line interpolation. The purpose of this thesis is to investigate the extent of the superiority of parabolic interpolation over straight line interpolation and cubic interpolation over parabolic interpolation. It is the objective to do this by means of theorems, equations of existing formulas, and by compiled tables.

This study is limited in scope to three methods of interpolation as applied to the sine function between zero and ninety degrees. The methods of interpolation used are three: straight line interpolation, parabolic interpolation, and cubic interpolation. The table developed for this investigation is taken at four different points of the range. Values are obtained near zero, thirty, sixty, and ninety degrees. The widths of the intervals used are constant, meaning that the control points are evenly spaced. Equal intervals are used because mathematical tables give trigonometric functions at uniform intervals.

The trigonometric tables given in most textbooks and in mathematical handbooks vary in length and accuracy from three to seven decimal places. A table to fifteen decimal places at hundredths of a degree was used to compile the table of interpolated values for this investigation.[1] The fifteen place table was used so that an accurate comparison could be obtained. The superiority of parabolic interpolation over straight line interpolation and of cubic interpolation over parabolic interpolation could not be

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[1] <u>Table of Sines and Cosines to Fifteen Decimal Places at Hundredths of a Degree</u> (Washington, D. C.: United States Government Printing Office, 1949), pp. 2-95.

secured from an ordinary seven place handbook table. Evidence that one method of interpolation was more accurate than another did not appear at times until the sixth decimal place while at other times this occured at the twelfth decimal point. In straight line interpolation two control points, a_0 and a_1, are needed. For parabolic interpolation three control points, a_0, a_1, and a_2, are needed. In cubic interpolation four control points are used, a_0, a_1, a_2, and a_3.

The width of the interval between points used are Δa equal to one degree and Δa equal to one-tenth degree. Interpolation could be performed with the width of the interval Δa as one-hundredth degree obtaining values such as sin 30.007 degrees but since the true value of sin 30.007 degrees is not given in the table and it is impossible to determine the error, Δa equals .01 degree is not used.

One of the most famous of the early printed tables of the trigonometric functions is the Opu Palatinum, compiled by the German mathematician Rhaeticus and published in 1596. This table gives the values of trigonometric functions to ten decimal places for angles spaced at intervals of ten seconds. It is often said that the work of Rhaeticus has never been superseded.[2]

The sample of points selected for the master table in this study is chosen so an idea of interpolation as applied to different parts of the sine curve can be secured. The sine of zero degrees

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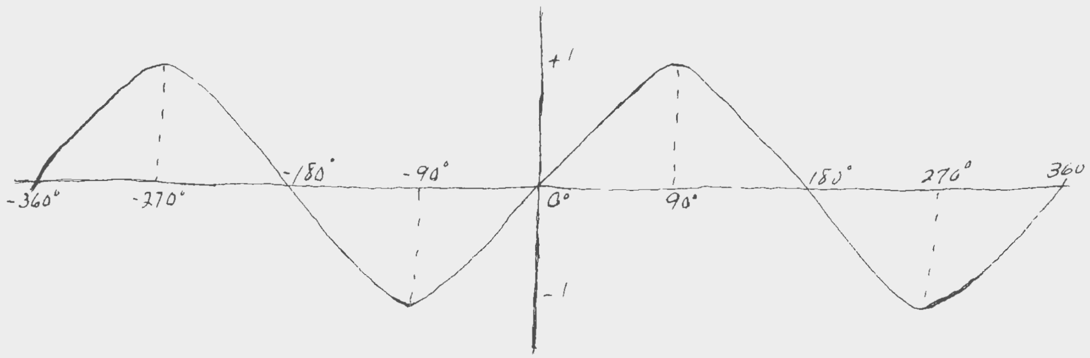
[2] William L. Hart, Plane Trigonometry with Applications (Boston: D. C. Heath and Company, 1942), p. 14.

is zero and gradually increases in value to one at ninety degrees. The graph of the sine function consists of infinitely many repetitions, to the right and to the left, of that part of the graph obtained as the angle varies from zero to 360 degrees. In particular, one observes that the graph of sin x consists of many identical waves. The graph of y = sin x consists of congruent segments 90 degrees in width.



y = sin x

Evidence reveals that extensive related research on interpolation exists. This is indicated by the number of different formulas available for interpolation. No immediate research, however, has been found relating to different methods of interpolation as applied to the sine function.

UNIT I

There are a number of formulas for interpolation, the Newton-Gauss formula of interpolation, the Newton-Sterling formula of interpolation, the Newton-Bessel formula of interpolation, and Lagrange's formula of interpolation, to list a few. The Lagrange's formula of interpolation was chosen for this study. Lagrange's formula is easily aplied to the sine function when using straight line, parabolic, and cubic methods of interpolation. In the Lagrange's general formula a polynomial of any degree may be used, that is f $(a_0, a_1, a_2, \ldots, a_n)$. It can also be used for any interval, whether or not the intervals are equally spaced being of little importance. However in this study the intervals Δa are always taken as of uniform length.

The two control points for straight line interpolation $\overline{\sin}\, x$ are designated by a_0 and a_1, and the difference is denoted by Δa. To illustrate, if $\Delta a = 1$ degree the control points are $a_0 = 0^\circ$ and $a_1 = 1^\circ$ when interpolating for $x = .1^\circ$, $.3^\circ$, $.5^\circ$, $.7^\circ$, or $.9^\circ$.

To obtain the sines of these same angles by parabolic interpolation, $\overset{\smile}{\sin}\, x$, three control points are necessary, a_0, and a_1, and a_2. Δa is the difference between two adjacent control points, or $a_1 - a_0 = a_2 - a_1$. The notation when placing the wanted values between the first two points a_0 and a_1 is indicated $\overset{\smile}{\sin}_1\, x$ and when placed between a_1 and a_2 is indicated $\overset{\smile}{\sin}_2\, x$. That is, when the first interval of parabolic interpolation is used, one point, a_0, would be to the left and the other two points, a_1 and a_2, to the right of the desired value. When second interval parabolic interpolation $\overset{\smile}{\sin}_2 x$ is used, two points a_0 and a_1, would be to the left and one point, a_2, to the right of the value wanted. To illustrate, for $\overset{\smile}{\sin}_1 x$, $\Delta a = 1^\circ$, $a_0 = 0^\circ$, $a_1 = 1^\circ$, and $a_2 = 2^\circ$ and the interpolation would be for $x = .1^\circ$, $.3^\circ$, $.5^\circ$, $.7^\circ$, and $.9^\circ$ using first interval parabolic interpolation. This would change when interpolating for the second interval parabolic value, $\overset{\smile}{\sin}_2 x$, $\Delta a = 1^\circ$, $a_{\cdot} = 1^\circ$, $a_0 = -1^\circ$, $a_1 = 0^\circ$, and $a_2 = 1^\circ$, while $x = .1^\circ$, $.3^\circ$, $.5^\circ$, $.7^\circ$, and $.9^\circ$.

The four control points for cubic interpolation, $\widetilde{\sin}\, x$, a_0, a_1, a_2, and a_3, are always taken the same distance apart so that Δa is the difference $a_1 - a_0$, $a_2 - a_1$, or $a_3 - a_2$. The notation when placing wanted values between a_0 and a_1 is indicated $\widetilde{\sin}_1 x$, when placed between a_1 and a_2 is indicated $\widetilde{\sin}_2 x$, and when placed between a_2 and a_3 is indicated $\widetilde{\sin}_3 x$.

When first interval cubic interpolation $\widetilde{\sin}_1 x$ is used one point a_0 is to the left and the other three points a_1, a_2, and a_3, to the right of the value wanted. When second interval cubic interpolation $\widetilde{\sin}_2 x$ is used, two points, a_0 and a_1 are to the left and the other two points, a_2 and a_3, to the right of the value wanted. When third interval cubic interpolation, $\widetilde{\sin}_3 x$, is used, three points, a_0, a_1, and a_2, are to the left and the and the other point, a_3, to the right.

To illustrate, for $\widetilde{\sin}_1 x$, $\Delta a = 1^\circ$, $a_o = 0^\circ$, $a_1 = 1^\circ$, $a_2 = 2^\circ$, and $a_3 = 3^\circ$ and the interpolation is for $x = .1^\circ$, $.3^\circ$, $.5^\circ$, $.7^\circ$, and $.9^\circ$. When interpolation is for second interval of cubic interpolation, $\widetilde{\sin}_2 x$, $\Delta a = 1^\circ$, $a_o = -1^\circ$, $a_1 = 0^\circ$, $a_2 = 1^\circ$, and $a_3 = 2^\circ$, with again $x = .1^\circ$, $.3^\circ$, $.5^\circ$, $.7^\circ$, and $.9^\circ$. Third interval cubic interpolation, $\widetilde{\sin}_3 x$, has $\Delta a = 1^\circ$, $a_o = -2^\circ$, $a_1 = -1^\circ$, $a_2 = 0^\circ$, and $a_3 = 1^\circ$ when $x = .1^\circ$, $.3^\circ$, $.5^\circ$, $.7^\circ$, and $.9^\circ$.

The difference between the true value and the interpolated values, which is the error, is designated. $E(\bar{x})$ for straight line, $E_1(\breve{x})$ or $E_2(\breve{x})$ for parabolic interval interpolation, and $E_1(\tilde{x})$, $E_2(\tilde{x})$, or $E_3(\tilde{x})$ for cubic interpolation.

Lagrange's General Interpolation Formula:

Let $f(x)$ be the polynomial of degree n which for values a_o, a_1, a_2, a_3, ..., a_n of the argument a has values $f(a_o)$, $f(a_1)$, $f(a_2)$, ..., $f(a_n)$ respectively.[3] By method of divided difference, we have $f(a_o, a_1, a_2, \ldots, a_n)$

$$= \frac{f(x)}{(x - a_o)(x - a_1) \ldots (x - a_n)} + \frac{f(a_o)}{(a_o - x)(a_o - a_1)(a_o - a_n)} +$$

$$\frac{f(a_1)}{(a_1 - x)(a_1 - a_o) \ldots (a_1 - a_n)} + \ldots +$$

$$\frac{f(a_n)}{(a_n - x)(a_n - a_o) \ldots (a_n - a_{n-1})}$$

---

[3] Whittaker and Robinson, <u>A Short Course in Interpolation</u> (New York: D. Van Nostrand Company, 1923), pp. 28-29.

Another way of writing this formula is:

$$f(x) = \frac{(x - a_1)(x - a_2)\ \dots\ (x - a_n)}{(a_o - a_1)(a_o - a_2)\ \dots\ (a_o - a_n)}\ f(a_o) +$$

$$\frac{(x - a_o)(x - a_2)\ \dots\ (x - a_n)}{(a_1 - a_o)(a_1 - a_2)\ \dots\ (a_1 - a_n)}\ f(a_1) + \dots +$$

$$\frac{(x - a_o)(x - a_1)\ \dots\ (x - a_{n-1})}{(a_n - a_o)(a_n - a_1)\ \dots\ (a_n - a_{n-1})}\ f(a_n)$$

This formula specialized for the straight line involves only two points a_o and a_1.

$$f(\overline{x}) = \frac{(x - a_1)}{(a_o - a_1)}\ f(a_o) + \frac{(x - a_o)}{(a_1 - a_o)}\ f(a_1).$$

This formula applied to the sine function becomes:

$$\overline{\sin}\ x = \frac{(x - a_1)}{(a_o - a_1)}\ \sin a_o + \frac{(x \quad a_o)}{(a_1 - a_o)}\ \sin a_1.$$

Lagrange's formula as applied to parabolic interpolation, since three points, namely a_o, a_1, and a_2, are involved, becomes:

$$f(\breve{x}) = \frac{(x - a_1)(x - a_2)}{(a_o - a_1)(a_o - a_2)}\ f(a_o) + \frac{(x - a_o)(x - a_2)}{(a_1 - a_o)(a_1 - a_2)}\ f(a_1) +$$

$$\frac{(x - a_o)(x - a_1)}{(a_2 - a_o)(a_2 - a_1)}\ f(a_2)$$

This formula as applied to the sine function:

$$\widetilde{\sin x} = \frac{(x - a_1)(x - a_2)}{(a_o - a_1)(a_o - a_2)} \sin a_o + \frac{(x - a_o)(x - a_2)}{(a_1 - a_o)(a_1 - a_2)} \sin a_1 +$$

$$\frac{(x - a_o)(x - a_1)}{(a_2 - a_o)(a_2 - a_1)} \sin a_2$$

Lagrange's formula as applied to cubic interpolation, since four points, a_o, a_1, a_2, and a_3, are needed, is:

$$f\widetilde{(x)} = \frac{(x - a_1)(x - a_2)(x - a_3)}{(a_o - a_1)(a_o - a_2)(a_o - a_3)} f(a_o) +$$

$$\frac{(x - a_o)(x - a_2)(x - a_3)}{(a_1 - a_o)(a_1 - a_2)(a_1 - a_3)} f(a_1) +$$

$$\frac{(x - a_o)(x - a_1)(a - a_3)}{(a_2 - a_o)(a_2 - a_1)(a_2 - a_3)} f(a_2) +$$

$$\frac{(x - a_o)(x - a_1)(x - a_2)}{(a_3 - a_o)(a_3 - a_1)(a_3 - a_2)} f(a_3).$$

This formula as applied to the sine function becomes:

$$\widetilde{\sin x} = \frac{(x - a_1)(x - a_2)(x - a_3)}{(a_o - a_1)(a_o - a_2)(a_o - a_3)} \sin a_o +$$

$$\frac{(x - a_o)(x - a_2)(x - a_3)}{(a_1 - a_o)(a_1 - a_2)(a_1 - a_3)} \sin a_1 +$$

$$\frac{(x - a_o)(x - a_1)(x - a_3)}{(a_2 - a_o)(a_2 - a_1)(a_2 - a_3)} \sin a_2 +$$

$$\frac{(x - a_o)(x - a_1)(x - a_2)}{(a_3 - a_o)(a_3 - a_1)(a_3 - a_2)} \sin a_3$$

Some of the interpolated values were computed by the use of these Lagrange's formulas. An alternative form, using finite differences, proved somewhat easier to apply and was used for much of the computation. These formulas take on the following form:

$$\overline{\sin}\, x = y + \theta \Delta y$$

$$\breve{\sin}_1\, x = \overline{\sin}\, x - \frac{\theta(1 - \theta)}{2} \Delta y^2$$

$$\breve{\sin}_2\, x = \overline{\sin}\, x - \frac{\theta(1 - \theta)}{2} \Delta y^2$$

$$\widetilde{\sin}_1\, x = \breve{\sin}_1\, x + \frac{\theta(1 - \theta)(2 - \theta)}{6} \Delta y^3$$

$$\widetilde{\sin}_2\, x = \breve{\sin}_1\, x + \frac{\theta(1 - \theta)(2 - \theta)}{6} \Delta^3 y$$

Or

$$\widetilde{\sin}_2\, x = \breve{\sin}_2\, x - \frac{\theta(1 - \theta)(1 \quad \theta)}{6} \Delta^3 y$$

$$\widetilde{\sin}_3\, x = \breve{\sin}_2\, x - \frac{\theta(1 - \theta)(1 \quad \theta)}{6} \Delta^3 y$$

Here θ refers to the proportionate distance of x through the interval Δ a. Thus using $\Delta a = .1^\circ$, θ in the interpolation for sin 30.03 would equal .3, Δy, Δy^2, and Δy^3 is the customary notation for finite differences. Two finite differences of the same order, which look alike in the formulas, are not necessarily equal but are determined by the particular set of points used as control points, which in turn is determined by the values of x and the subscript used with the $\breve{\sin}\, x$ and $\widetilde{\sin}\, x$.

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TABLE I

x	sin x	$\overline{\sin x}$
.01	.00017 45329 24313	.00017 45328 36590
.03	.00052 35987 51674	.00052 35985 09769
.05	.00087 26645 15235	.00087 26641 82949
.07	.00122 17301 72465	.00122 17298 56129
.09	.00157 07956 80831	.00157 07955 29308
30.01	.50015 11423 30817	.50015 10737 15946
30.03	.50045 33812 81421	.50045 32211 47837
30.05	.50075 55592 53300	.50075 53685 79728
30.07	.50105 76762 09632	.50105 75160 11620
30.09	.50135 97321 13606	.50135 96634 43511
60.01	.86611 26570 56274	.86611 25382 99555
60.03	.86628 70844 47387	.86628 68073 29788
60.05	.86646 14062 84047	.86646 10763 60021
60.07	.86663 56225 45013	.86663 53453 90254
60.09	.86680 97332 09057	.86680 96144 20487
89.01	.99985 07259 47372	.99985 05888 88918
89.03	.99985 66961 57657	.99985 63763 53972
89.05	.99986 25445 38437	.99986 21638 19026
89.07	.99986 82710 88999	.99986 79512 84080
89.09	.99987 38758 08645	.99987 37387 49134

$\Delta a = .1$ degree

TABLE I (Continued)

x	$\breve{\sin}_1\ x$	$\breve{\sin}_2\ x$
.01	.00017 45330 75836	.00017 45328 36590
.03	.00052 35990 68009	.00052 35985 09769
.05	.00087 26648 47521	.00087 26641 82949
.07	.00122 17304 14369	.00122 17298 56129
.09	.00157 17957 68554	.00157 17955 29308
30.01	.50015 11424 61937	.50015 11422 54848
30.03	.50045 33815 55148	.50045 33810 71942
30.05	.50075 55595 40813	.50075 55589 65567
30.07	.50105 76764 18931	.50105 76759 35724
30.09	.50135 97321 89502	.50135 97319 82414
60.01	.86611 26571 01858	.86611 26570 12416
60.03	.86628 70846 05161	.86628 70843 26463
60.05	.86646 14064 49751	.86646 14061 17967
60.07	.86663 56276 65627	.86663 56223 86929
60.09	.86680 97332 52790	.86680 97331 33348
89.01	.99985 07259 49811	.99985 07259 45844
89.03	.99985 66961 62723	.99985 66961 53467
89.05	.99986 25445 43729	.99986 25445 32710
89.07	.99986 82710 92831	.99986 82710 83575
89.09	.99987 38758 10027	.99987 38758 06060

Δ a = .1 degree

TABLE I (Continued)

x	$\widetilde{\sin_1} x$	$\widetilde{\sin_2} x$	$\widetilde{\sin_3} x$
.01	.00017 45329 24314	.00017 45329 24313	.00017 45329 24313
.03	.00052 35987 51674	.00052 35987 51673	.00052 35987 51674
.05	.00087 26645 15236	.00087 26645 15235	.00087 26645 15236
.07	.00122 17301 72466	.00122 17301 72465	.00122 17301 72465
.09	.00157 07956 80831	.00157 07956 80830	.00157 07956 80831
30.01	.50015 11423 30914	.50015 11423 30781	.50015 11423 30857
30.03	.50045 33812 81608	.50045 33812 81331	.50045 33812 81543
30.05	.50075 55592 53481	.50075 55592 53190	.50075 55592 53480
30.07	.50105 76762 09753	.50105 76762 09542	.50105 76762 09817
30.09	.50135 97321 13646	.50135 97321 13570	.50135 97321 13701
60.01	.86611 26570 56439	.86611 26570 56211	.86611 26570 56344
60.03	.86628 70844 47711	.86628 70844 47232	.86628 70844 47598
60.05	.86646 14062 84362	.86646 14062 83859	.86646 14062 84361
60.07	.86663 56225 45224	.86663 56225 44858	.86663 56225 45336
60.09	.86680 97332 09127	.86680 97332 08995	.86680 97332 09224
89.01	.99985 07259 47563	.99985 07259 47299	.99985 07259 47453
89.03	.99985 66961 58030	.99985 66961 57478	.99985 66961 57900
89.05	.99986 25445 38799	.99986 25445 38220	.99986 25445 38799
89.07	.99986 82710 89242	.99986 82710 88820	.99986 82710 89372
89.09	.99987 38758 08726	.99987 38758 08573	.99987 38758 08837

$\Delta a =$ .1 degree

TABLE I (Continued)

.x	sin x	$\overline{\sin}\ x$
.1	.00174 53283 65898	.00174 52406 43728
.3	.00523 59638 31420	.00523 57219 31185
.5	.00872 65354 98374	.00872 62032 18642
.7	.01221 70008 35247	.01221 66845 06099
.9	.01570 73173 11821	.01570 71657 93556
30.1	.50151 07371 59457	.50150 38074 91005
30.3	.50452 76238 15019	.50451 14224 73016
30.5	.50753 83629 60704	.50751 90374 55027
30.7	.51054 29179 11606	.51052 66524 37038
30.9	.51354 12520 58170	.51353 42674 19049
60.1	.86689 67489 35603	.86668 48341 19935
60.3	.86863 15144 38191	.86860 36947 90926
60.5	.87035 56959 39900	.87032 25554 61918
60.7	.87206 92724 32131	.87204 14161 32090
60.9	.87377 22230 35465	.87376 02768 03900
89.1	.99987 66324 81561	.99986 29256 40752
89.3	.99992 53696 60452	.99989 33866 09474
89.5	.99996 19230 64171	.99992 38475 78195
89.7	.99998 62922 47427	.99995 43085 46917
89.9	.99999 84769 13288	.99998 47695 15639

$\Delta a =$ 1 degree

TABLE I (Continued)

$.x$	$\breve{\sin}_1 x$	$\breve{\sin}_2 x$
.1	.00174 54798 71471	.00174 52406 43728
.3	.00523 62801 29801	.00523 57219 31185
.5	.00872 68677 40150	.00872 62032 13642
.7	.01221 72427 04166	.01221 66845 06099
.9	.01570 74050 21299	.01570 71657 93556
30.1	.50151 08673 42416	.50151 06612 08967
30.3	.50452 78954 59641	.50452 74144 81595
30.5	.50753 86481 53389	.50753 80755 60478
30.7	.51054 31254 23662	.51054 26444 45617
30.9	.51354 13272 70460	.51354 11211 37011
60.1	.86689 68229 13529	.86689 67050 17667
60.3	.86863 16686 42646	.86863 13935 52300
60.5	.87035 58576 66346	.87035 55301 77839
60.7	.87206 93899 84629	.87206 91148 94283
60.9	.87377 22655 97494	.87377 21427 01632
89.1	.99987 66330 76677	.99987 66309 88968
89.3	.99992 53706 26632	.99992 53657 55311
89.5	.99996 19237 89097	.99996 19179 89906
89.7	.99998 62925 64075	.99998 62876 92754
89.9	.99999 84769 51564	.99999 84748 63855

$\Delta a = 1$ degree

TABLE I (Continued)

.x	$\widetilde{\sin}_1 x$	$\widetilde{\sin}_2 x$	$\widetilde{\sin}_3 x$
.1	.00174 53284 06719	.00174 53283 60567	.00174 53283 60567
.3	.00523 59639 13366	.00523 59638 17014	.00523 59638 17014
.5	.00872 65355 80606	.00872 65354 79396	.00872 65354 79396
.7	.01221 70008 92018	.01221 70008 18337	.01221 70008 18337
.9	.01570 73173 31179	.01570 73173 04460	.01570 73173 04460
30.1	.50151 07381 53214	.50151 07367 91232	.50151 07375 56725
30.3	.50452 76257 48852	.50452 76229 05415	.50452 76250 16321
30.5	.50753 83648 43738	.50753 83618 50933	.50753 83647 56530
30.7	.51054 29191 74236	.51054 29169 99842	.51054 29197 60258
30.9	.51354 12524 76711	.51354 12516 88195	.51354 12530 10411
60.1	.86689 67506 44515	.86689 67482 13150	.86689 67495 39007
60.3	.86863 15177 65232	.86863 15126 67117	.86863 15163 23267
60.5	.87035 56991 81067	.87035 56937 97093	.87035 56988 19277
60.7	.87206 92746 07783	.87206 92700 59812	.87206 92754 40932
60.9	.87377 22237 57539	.87377 22223 12012	.87377 22246 02128
89.1	.99987 66343 98893	.99987 66317 54461	.99987 66332 85215
89.3	.99992 53733 87047	.99992 53678 66217	.99992 53720 87385
89.5	.99996 19266 88693	.99996 19208 89501	.99996 19266 67810
89.7	.99998 62946 74981	.99998 62904 53169	.99998 62959 73158
89.9	.99999 84777 17057	.99999 84761 86071	.99999 84788 30099

$\Delta a = 1$ degree

TABLE II

x	$\sin x - \overline{\sin}\, x$	$\breve{\sin}_1 x - \sin x$	$\sin x - \breve{\sin}_2 x$
.01	.00000 00000 87723	.00000 00001 51523	.00000 00000 87723
.03	.00000 00002 41905	.00000 00003 16335	.00000 00002 41905
.05	.00000 00003 32286	.00000 00003 32286	.00000 00003 32286
.07	.00000 00003 16336	.00000 00002 41904	.00000 00003 16336
.09	.00000 00001 51523	.00000 00000 87723	.00000 00001 51523
30.01	.00000 00686 14871	.00000 00001 31120	.00000 00000 75969
30.03	.00000 01601 33584	.00000 00002 73727	.00000 00002 09479
30.05	.00000 01906 73572	.00000 00002 87513	.00000 00002 87733
30.07	.00000 01601 98012	.00000 00002 09299	.00000 00002 73908
30.09	.00000 00686 70095	.00000 00000 75696	.00000 00001 31193
60.01	.00000 01187 56719	.00000 00000 75584	.00000 00000 43858
60.03	.00000 02771 17599	.00000 00001 57774	.00000 00001 20924
60.05	.00000 03299 24026	.00000 00001 65704	.00000 00001 66080
60.07	.00000 02771 54759	.00000 00001 20614	.00000 00001 58084
60.09	.00000 01187 88570	.00000 00000 43733	.00000 00000 75709
89.01	.00000 01370 58454	.00000 00000 02439	.00000 00000 01528
89.03	.00000 03198 03685	.00000 00000 05066	.00000 00000 04190
89.05	.00000 03807 19411	.00000 00000 05292	.00000 00000 05727
89.07	.00000 03198 04919	.00000 00000 03832	.00000 00000 05424
89.09	.00000 01370 59511	.00000 00000 01382	.00000 00000 02585

TABLE II (Continued)

x	$\sin x - \overline{\sin}\, x$	$\breve{\sin}_1 x - \sin x$	$\sin x - \breve{\sin}_2 x$
.1	.00000 00877 22170	.00000 01515 05573	.00000 00877 22170
.3	.00000 02419 00235	.00000 03162 97832	.00000 02419 00235
.5	.00000 03322 79732	.00000 03322 41776	.00000 03322 79732
.7	.00000 03163 29148	.00000 02417 68919	.00000 03163 29148
.9	.00000 01515 18275	.00000 00877 09478	.00000 01515 18275
30.1	.00000 69296 68452	.00000 01301 82959	.00000 00759 50490
30.3	.00001 62013 42003	.00000 02716 44622	.00000 02093 33424
30.5	.00001 93255 05677	.00000 02851 92685	.00000 02874 00226
30.7	.00001 62654 74568	.00000 02075 12056	.00000 02734 65989
30.9	.00000 69846 39121	.00000 00752 12290	.00000 01309 21159
60.1	.00001 19148 15668	.00000 00739 77926	.00000 00439 17936
60.3	.00002 78196 47265	.00000 01542 04455	.00000 01208 85891
60.5	.00003 31404 77982	.00000 01617 26446	.00000 01657 62061
60.7	.00002 78562 99212	.00000 01175 52508	.00000 01575 37838
60.9	.00001 19462 31565	.00000 00425 62029	.00000 00803 33833
89.1	.00001 37068 40909	.00000 00005 95016	.00000 00014 92693
89.3	.00003 19830 50976	.00000 00009 66180	.00000 00039 05141
89.5	.00003 80754 85976	.00000 00007 24926	.00000 00050 74265
89.7	.00003 19837 00510	.00000 00003 16648	.00000 00045 54673
89.9	.00001 37073 97649	.00000 00000 38276	.00000 00020 49433

TABLE II (Continued)

x	$\widetilde{\sin_1 x} - \sin x$	$\sin x - \widetilde{\sin_2 x}$	$\widetilde{\sin_3 x} - \sin x$
.01	.00000 00000 00001	.00000 00000 00000	.00000 00000 00000
.03	.00000 00000 00000	.00000 00000 00000	.00000 00000 00000
.05	.00000 00000 00001	.00000 00000 00000	.00000 00000 00001
.07	.00000 00000 00001	.00000 00000 00000	.00000 00000 00001
.09	.00000 00000 00000	.00000 00000 00001	.00000 00000 00000
30.01	.00000 00000 00097	.00000 00000 00036	.00000 00000 00040
30.03	.00000 00000 00187	.00000 00000 00090	.00000 00000 00122
30.05	.00000 00000 00181	.00000 00000 00110	.00000 00000 00180
30.07	.00000 00000 00121	.00000 00000 00090	.00000 00000 00185
30.09	.00000 00000 00040	.00000 00000 00036	.00000 00000 00095
60.01	.00000 00000 00165	.00000 00000 00063	.00000 00000 00070
60.03	.00000 00000 00324	.00000 00000 00155	.00000 00000 00211
60.05	.00000 00000 00315	.00000 00000 00198	.00000 00000 00314
60.07	.00000 00000 00211	.00000 00000 00155	.00000 00000 00323
60.09	.00000 00000 00070	.00000 00000 00062	.00000 00000 00167
89.01	.00000 00000 00191	.00000 00000 00073	.00000 00000 00081
89.03	.00000 00000 00373	.00000 00000 00179	.00000 00000 00243
89.05	.00000 00000 00362	.00000 00000 00217	.00000 00000 00362
89.07	.00000 00000 00243	.00000 00000 00179	.00000 00000 00373
89.09	.00000 00000 00081	.00000 00000 00072	.00000 00000 00192

TABLE II (Continued)

x	$\widetilde{\sin_1} x - \sin x$	$\sin x - \widetilde{\sin_2} x$	$\widetilde{\sin_3} x - \sin x$
.1	.00000 00000 40821	.00000 00000 05331	-.00000 00000 05331
.3	.00000 00000 81946	.00000 00000 14406	-.00000 00000 14406
.5	.00000 00000 82232	.00000 00000 18978	-.00000 00000 18978
.7	.00000 00000 56771	.00000 00000 16910	-.00000 00000 16910
.9	.00000 00000 19358	.00000 00000 07361	-.00000 00000 07361
30.1	.00000 00009 93757	.00000 00003 68225	.00000 00003 97268
30.3	.00000 00019 33833	.00000 00009 09604	.00000 00012 01301
30.5	.00000 00018 83034	.00000 00011 03771	.00000 00017 95826
30.7	.00000 00012 62630	.00000 00009 11764	.00000 00018 48652
30.9	.00000 00004 18541	.00000 00003 69975	.00000 00009 52241
60.1	.00000 00017 08912	.00000 00007 22453	.00000 00006 03404
60.3	.00000 00033 27041	.00000 00017 71074	.00000 00018 85076
60.5	.00000 00032 41707	.00000 00021 42807	.00000 00028 79377
60.7	.00000 00021 35661	.00000 00017 72309	.00000 00030 08811
60.9	.00000 00007 22074	.00000 00007 23453	.00000 00015 66663
89.1	.00000 00019 17232	.00000 00007 27200	.00000 00008 03554
89.3	.00000 00037 26595	.00000 00017 94235	.00000 00024 26933
89.5	.00000 00036 24522	.00000 00021 74670	.00000 00036 23639
89.7	.00000 00024 27554	.00000 00017 94258	.00000 00037 25731
89.9	.00000 00008 03769	.00000 00007 27217	.00000 00019 16811

UNIT III

STRAIGHT LINE INTERPOLATION

One of the first things noted from an examination of Table II is that $\sin x - \overline{\sin}\, x$ is always positive. This means that the straight line interpolated value always is less than the true value. This would be expected by anyone familiar with the sine curve. A proof of this more or less obvious fact is presented later.[4]

It is seen from the table that with each of the kinds of interpolation the error of interpolation tends to be less nearer the control points, reaching the maximum in the case of the straight line someplace near the midpoint of the interpolation interval. This, likewise, probably would have been anticipated.

The error using straight line interpolation is seen to vary as x takes different values from near zero to near ninety degrees. The accuracy is greatest near zero degrees where the curvature of the sine is least, and decreases, generally speaking, as the angle increases toward ninety degrees.

The width of the interpolation interval has considerable bearing on the accuracy of straight line interpolation. The amount of accuracy of straight line interpolation is increased at least a hundredfold as the interval of interpolation is

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[4] See the appendix page 30.

divided by ten. This would be expected as the closer together the control points are placed, the nearer the straight line approaches the sine curve.

PARABOLIC INTERPOLATION

In considering parabolic interpolation, as well as in making comparisons between it and straight line interpolation, it is necessary to distinguish between that which has been designated by $\breve{\sin}_1 x$ and $\breve{\sin}_2 x$. The values $\breve{\sin}_1 x$ are always greater than sin x and the values $\breve{\sin}_2 x$ are always less.[5]

While there still remains the previously mentioned tendency of the error for different values of x within the interpolation interval to increase to a maximum somewhere near the center of the interval and then decrease again, a skewness is observed. There is a tendency, found by comparing values of $E_1(\breve{x})$ with $E_2(\breve{x})$, for the $\breve{\sin}_1 x$ or $\breve{\sin}_2 x$ to be the more accurate which results in the interpolated value of x being nearer the middle interpolation point. In fact, for points near a control point, choosing $\breve{\sin}_1 x$ or $\breve{\sin}_2 x$ according to this principle approximately divides the error by two.

Some values of $E_1(\breve{x})$ are greater than $E_2(\breve{x})$ and some are less. They average about the same but the average $E_1(\breve{x})$ is less, very slightly, than the average $E_2(\breve{x})$. This difference results from the way in which the slope and curvature of the sine curve vary through first quadrant values of the angle.

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[5] See the appendix page 31.

$E_1(\breve{x})$ and $E_2(\breve{x})$, unlike $E(\overline{x})$, tend to decrease as the angle increases from zero to ninety degrees. Just as straight line interpolation tends to be better near zero degrees where the sine curve approaches a straight line, parabolic interpolation tends to be more accurate near ninety degrees where the sine curve is more like a part of a parabola.

A decrease in the width of the interpolation interval improves the accuracy of parabolic interpolation to an even greater degree than it did in the case of straight line interpolation. In straight line interpolation when the interval was divided by ten, it improve the accuracy by two decimal places; the parabolic interpolation does even better, being more accurate to three decimal points. When comparing the curves, the sine curve and the parabola, the nearer the control points are placed together, the more they tend to coincide.

Parabolic interpolation, both $\breve{\sin}_1 x$ and $\breve{\sin}_2 x$, prove to be far more accurate than $\overline{\sin}\, x$ for almost all values of x. Values of x very near zero degrees, using zero as a control point, proves an interesting exception to the general rule. The values $\breve{\sin}_2 x$ exactly equal $\overline{\sin}\, x$ in certain cases. In these cases, x is a first quadrant angle but the control points are $-\Delta a$, 0, and $+\Delta a$. Since $\sin(-x)$ equals $-\sin x$, it so happens that the "parabola" through $(-\Delta a, \sin -\Delta a)$, $(0, \sin 0)$, and $(\Delta a, \sin \Delta a)$ is in reality a straight line. Surprisingly, using control points $(0, \Delta a$, and $2\Delta a)$, $\breve{\sin}_1 x$ proves to be less accurate than $\overline{\sin}\, x$ when x is in the left half, approximately, of the interpolation interval. This might be explained on the grounds that the sine curve is so very nearly straight in the vicinity of zero degrees that straight line interpolation actually proves better at this point.

The difference, $\overline{E}\,x - \breve{E}_1 x$ or $\overline{E}\,x - \breve{E}_2 x$, might be called the superiority of parabolic interpolation over straight line interpolation. In both cases this superiority seems to reach its maximum about midway through the interpolation interval. In the case of $\breve{\sin}_2 x$ this point of maximum superiority is exactly at the midpoint.[6]

CUBIC INTERPOLATION

By comparing the three different cubic interpolations for the same value of x, $\widetilde{\sin}_1 x$, $\widetilde{\sin}_2 x$, and $\widetilde{\sin}_3 x$ with the correct values of sin x it is seen that $\widetilde{\sin}_1 x$ is always greater than the true value, $\widetilde{\sin}_2 x$ is always less, and $\widetilde{\sin}_3 x$ is greater everywhere except for values of x near enough to zero degrees to require use of some angles that are negative angles for control points. $\widetilde{E}_2 x$ averages appreciably less than either $\widetilde{E}_1 x$ or $\widetilde{E}_3 x$ which are about equal. $\widetilde{E}_3 x$ is very slightly less on the average than $\widetilde{E}_1 x$.

Again the errors are small near the ends of the interpolation interval and attain a maximum value someplace within the interval

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[6] See the appendix page 32.

very close to the midpoint of the interval and the error decreases fairly symmetrically on each side of the maximum value. In the case of $\widetilde{E_1 x}$ and $\widetilde{E_3 x}$ this tendency of the error to reach a maximum midway in the interval is fairly well offset by the tendency of the error to be greater the farther the x is from the point midway between the two outside control points.

Cubic interpolation resembles linear interpolation rather than parabolic in that it decreases in accuracy as the angle increases. This might be explained by saying that the cubic, at least parts of it, tends to appear more nearly like this sine curve at near zero degree values.

The accuracy of cubic interpolation increases as the width of the interpolation interval decreases, to an even greater extent than is true with either straight line interpolation or parabolic interpolation. In straight line interpolation as the width of the interpolation interval is divided by ten, the accuracy is increased two decimal places. In parabolic interpolation the same decrease in the width of the interval increases the accuracy three decimal places, and in the case of cubic interpolation, the accuracy is increased four decimal places.

In practically every case the cubic interpolated values are more accurate than the straight line or the parabolic. The only exception occurs near ninety degrees, very near to ninety degrees. Then the cubic values are more accurate than the $\overline{\sin x}$ and

$\breve{\sin}_2 x$ but less accurate than the $\breve{\sin}_1 x$. When this takes place, x is a first quadrant angle but is so close to ninety degrees that some of the control points are second quadrant angles.

SUMMARY

A study of this investigation reveals that the results obtained are somewhat different than anticipated.

Cubic interpolation proves to be the most accurate of the three methods used, with one exception. This exception occurs near ninety degrees when values greater than ninety are used as control points.

Parabolic interpolation proves to be more accurate than straight line interpolation, with one exception. The exception occurs when zero degrees is used as the control point. Under this circumstance straight line interpolation proves just as accurate as second interval parabolic interpolation, and more accurate than first interval parabolic interpolation to nearly the mid-point of the interval. From there on, straight line interpolation is less accurate.

The most practical choice would be to use straight line interpolation. Under normal conditions, the extra work of computation involved for parabolic and cubic interpolation weighed against the improvement of accuracy would not seem justified.

The accuracy of interpolation depends upon other factors besides the method of interpolation. The tables, the number of decimal places and the width of the interval, are the other important factors. When using tables to five or seven decimal points there are no advantages in using parabolic or cubic methods of interpolation if the values are given whole degrees or parts of a degree.

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APPENDIX

THEOREM: No nth degree polynomial can meet $y = \sin x$ in more than $n + 1$ distinct points between $x = 0$ and $x = \pi/2$.

Proof: Assume the contrary, let $P_n(x)$ be an nth degree polynomial such that $f(x) = \sin x - P_n(x)$ vanishes for $n + 2$ distinct points x_i, $0 \leq x_1 \leq x_2 \ldots x_{n+2} \leq \pi/2$.

By Rolles' Theorem: If a function $f(x)$ and its derivative $f^1(x)$ are continuous for all values of x throughout an interval $0 \leq x \leq \pi/2$, there exists at least one value such $f^1(x) = 0$;[7] or $P_n^1(x)$, an $(n - 1)$ degree polynomial, must meet $y = \cos x$ in $n + 1$ distinct points between 0 and $\pi/2$.

Continuing: $f^k(x)$ must vanish for $n + 2 - k$ distinct points between 0 and $\pi/2$ or $P_n^k(x)$, an $n - k$ degree polynomial, must meet $y = \sin x$, $y = \cos x$, $y = - \sin x$, or $y = - \cos x$ in $n + 2 - k$ distinct points. If $k = n$, $P_n^k(x)$ which will be a constant, will meet $y = \sin x$, $y = \cos x$, $y = - \sin x$ or $y = - \cos x$ in two distinct points between 0 and $\pi/2$. This is contrary to our knowledge because no two distinct first quadrant angles have the same values for their sines or for their cosines. Thus the assumption is not correct and therefore the theorem is proved.

Lagranges' nth degree interpolation formula $L_n(x)$ provides

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[7] Edward S. Smith, Meyer Salkover, and Howard K. Justice, Calculus (New York: John Wiley and Sons, Inc., 1938), p. 333.

an nth degree polynomial meeting $y = \sin x$ in $n+1$ distinct points between $x = 0$ and $x = \pi/2$, $f(x) = a_o, a_1, a_2, \dots a_n$.

$$f(x) = \sin x - \left[\frac{(x - a_1)(x - a_2) \dots (x - a_n)}{(a_o - a_1)(a_o - a_2) \dots (a_o - a_n)} \sin a_o \right.$$

$$\left. + \dots + \frac{(x - a_o)(x - a_2) \dots (x - a_{n-1})}{(a_n - a_o)(a_n - a_2) \dots (a_n - a_{n-1})} \sin_n a \right].$$

$$\sin x - \sum_{i=o}^{n} \frac{(x - a_o) \dots (x - a_{i-1})(x - a_{i+1}) \dots (x - a_n)}{(a_i - a_o) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)} \sin a_i$$

$f(x) = 0$, for $x = a_o, a_1, a_2, \dots, a_n$ for all a_k since when $x = a_k$ all terms of the summation vanish except the one for which $i = k$. By previous theorem $L_n(x)$ will meet $y = \sin x$ in no more than $n+1$ distinct points $a_o, a_1, a_2, \dots, a_n$.

Thus $E(x)$, the error involved in using the interpolation formula $E(x) = \sin x - L_n(x)$ will vanish between 0 and $\pi/2$ only for $a_o, a_1, a_2, \dots, a_n$. If $E(x) > 0$ (or < 0) for $x = \dfrac{a_i + a_{i+1}}{2}$ it will be > 0 (or < 0) for all x's between a_i and a_{i+1}.

Corollary: If $P_n(x)$, an nth degree polynomial, meets $y = \sin x$ in $n+1$ distinct points between 0^o and 90^o, it will actually cross the sine curve at each of these points instead of being tangent.

THEOREM: The straight line interpolated value for $\overline{\sin x}$, x and the control points being in the first quadrant, is less than sin x for all angles between the control points.

Proof: $E(x) = f(x) - f(\overline{x}) > 0 \qquad a_o < x < a_1$

$$(1)\ \sin x - \frac{(x - a_1)}{(a_o - a_1)} \sin a - \frac{(x - a_1)}{(a_1 - a_o)} \sin a_1 > 0$$

Let $x = \frac{a_o + a_1}{2}$, and $\Delta a = a_1 - a_o$

This substitution gives:

$$(2)\quad \sin \frac{(a_o + a_1)}{2} - \frac{\frac{(a_o + a_1 - 2a_1)}{2}}{-\Delta a} \sin a_o - \frac{\frac{(a_o + a_1 - 2a_1)}{2}}{\Delta a} \sin a_1 > 0$$

Combining like terms, simplifying, and transposing the last two terms, the inequality (2) becomes:

$$(3)\quad \sin \frac{(a_o + a_1)}{2} > \frac{1}{2} \sin a_o + \frac{1}{2} \sin a_1$$

Using the trigonometric formula $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x - y}{2}$

the inequality becomes:

$$(4)\quad 2 \sin \frac{a_o + a_1}{2} > 2 \sin \frac{a_o + a_1}{2} \cos \frac{a_1 - a_o}{2}$$

Dividing both sides of the inequality by $2 \sin \frac{a_o + a_1}{2}$ (4)

becomes:

$$(5)\quad 1 > \cos \frac{a_1 - a_o}{2}$$

THEOREM: The parabolic interpolated value for sin x, x and the control points being in the first quadrant, $\breve{\sin}_1 x$ is greater than sin x and $\breve{\sin}_2 x$ is less than sin x for all angles between the control points.

Proof: $E(x) = f_1(\breve{x}) - f(x) > 0 \qquad a_o < x < a_1$

$$(1) \quad \frac{(x - a_1)(x - a_2)}{(a_o - a_1)(a_o - a_2)} \sin a_o + \frac{(x - a_o)(x - a_2)}{(a_1 - a_o)(a_1 - a_2)} \sin a_1 + \frac{(x - a_o)(x - a_1)}{(a_2 - a_o)(a_2 - a_1)} \sin a_2 - \sin x > 0$$

Let $\Delta a = a_1 - a_o = a_2 - a_1$

Substituting in the inequality (1), it becomes:

$$(2) \quad \frac{(x - a_1)(x - a_2)}{2 \Delta a^2} \sin a_o + \frac{(x - a_o)(x - a_2)}{- \Delta a^2} \sin a_1 + \frac{(x - a_o)(x - a_1)}{2 \Delta a^2} \sin a_2 - \sin x > 0$$

Let $x = \dfrac{a_o + a_1}{2}$

Upon substitution and simplification, inequality (2) becomes:

$$(3) \quad \frac{3}{8} \sin (a_1 - \Delta a) + \frac{3}{4} \sin a_1 - \frac{1}{8} \sin (a_1 + \Delta a) - \sin \frac{(2a_1 + \Delta a)}{2} > 0$$

Using the trigonometric formulas of the sum and the difference of two angles, inequality (3) becomes:

$$(4)\quad \frac{3}{8}\left(\sin a_1 \cos \Delta a - \cos a_1 \sin \Delta a\right) + \frac{3}{4}\sin a_1$$

$$- \frac{1}{8}\left(\sin a_1 \cos \Delta a + \cos a_1 \sin \Delta a\right) - \sin a_1 \cos \frac{\Delta a}{2}$$

$$+ \cos a_1 \sin \frac{\Delta a}{2} > 0$$

Combining like terms, inequality (4) becomes:

$$(5)\quad \frac{1}{4}\sin a_1 \cos \Delta a - \frac{1}{2}\cos a_1 \sin \Delta a + \frac{3}{4}\sin a_1 - \sin a_1 \cos \frac{\Delta a}{2}$$

$$+ \cos a_1 \sin \frac{\Delta a}{2} > 0$$

Factoring out $\sin a_1$ and $\cos a_1$, then rearranging terms, inequality (5) becomes:

$$(6)\quad \sin a_1 \left(\frac{1}{4}\cos \Delta a - \cos \frac{\Delta a}{2} + \frac{3}{4}\right) - \cos a_1 \left(\frac{1}{2}\sin \Delta a - \sin \frac{\Delta a}{2}\right) > 0$$

Using the double angle formula, $\sin \Delta a = 2 \sin \frac{\Delta a}{2} \cos \frac{\Delta a}{2}$

inequality (6) becomes:

$$(7)\quad \sin a_1 \left(\frac{3}{4} + \frac{1}{4}\cos \Delta a - \cos \frac{\Delta a}{2}\right) - \cos a_1 \sin \frac{\Delta a}{2}\left(\cos \frac{\Delta a}{2} - 1\right) > 0$$

The left side of the above inequality is greater than zero, as both terms are positive. Since the first interval parabolic interpolation stays above the sine curve, the second interval parabolic interpolation stays under the sine curve as indicated by the corollary.

THEOREM: When interpolating for sin x, x and the control points being in the first quadrant, the superiority of $\breve{\sin}_2 x$ over $\overline{\sin}\, x$ is greatest when x is midway between the two control points.

Proof: $E(\bar{x}) - E_2(\breve{x})$. This in terms of the functions is:

$(\sin x - \overline{\sin}\, x) - (\sin x - \breve{\sin}_2\, x)$

This becomes upon combining:

$\breve{\sin}_2 x - \overline{\sin}\, x$

Using the expression obtained for $\breve{\sin}_2 x$ in terms of $\overline{\sin}\, x$,

It becomes:

$$\overline{\sin}\, x - \frac{\theta(1 - \theta)}{2} \Delta^2 y - \overline{\sin}\, x$$

Combining like terms it becomes:

$$- \frac{\theta(1 - \theta)}{2} \Delta^2 y$$

Taking the derivative and equating to zero it becomes:

$$- 1 + 2\theta = 0$$

Solving this equation and obtaining $\theta = \frac{1}{2}$, the maximum benefit of second interval parabolic interpolation over straight line interpolation is secured.